Math 217.003 F25
Quiz 18 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose V is a vector space. The *dimension* of V is ...

   > **Solution:** The number of vectors in any basis of V (i.e., the common cardinality of all bases of V). If V has a finite basis with n vectors, then $\dim(V) = n$; if no finite basis exists, $\dim(V)$ is infinite.

   (b) An $n \times n$ matrix A is *invertible* provided that ...

   > **Solution:** There exists an $n \times n$ matrix A^{-1} such that $AA^{-1} = I_n$ and $A^{-1}A = I_n$; equivalently, the linear map $T_A : \mathbb{R}^n \to \mathbb{R}^n$ given by $T_A(x) = Ax$ is bijective (equivalently, $\det(A) \neq 0$, equivalently, $\operatorname{rank}(A) = n$).

   (c) Suppose X and Y are sets. A function $f : X \to Y$ is called *injective* provided that ...

   > **Solution:** For all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$; equivalently, if $x_1, x_2 \in X$ such that $x_1 \neq x_2$. then $f(x_1) \neq f(x_2)$.

2. Let $S : \mathbb{R}^m \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations such that for every $1 \leq i \leq m$, $(T \circ S)(\vec{e}_i) = \vec{e}_i$.

   (a) Prove that S is one-to-one (injective).

   > **Solution:** Since the standard basis $\{\vec{e}_1, \ldots, \vec{e}_m\}$ spans $\mathbb{R}^m$ and $T \circ S$ agrees with the identity on this basis, by linearity we have $(T \circ S)(x) = x$ for all $x \in \mathbb{R}^m$ as for any $x = c_1\vec{e}_1 + \cdots + c_n\vec{e}_n$, we obtain $(T \circ S)(x) = c_1(T \circ S)(\vec{e}_1) + \cdots + c_n(T \circ S)(\vec{e}_n) = c_1\vec{e}_1 + \cdots + c_n\vec{e}_n = x$. That is, $T \circ S = I_{\mathbb{R}^m}$. If $S(x) = S(y)$, then applying T gives $x = (T \circ S)(x) = (T \circ S)(y) = y$. Hence S is injective.

   (b) Prove that T is onto (surjective).

   > **Solution:** For any $w \in \mathbb{R}^m$, using $T \circ S = I_{\mathbb{R}^m}$ we have $w = (T \circ S)(w) = T(S(w))$. Thus w lies in the image of T. Since w was arbitrary, $\operatorname{im}(T) = \mathbb{R}^m$, so T is surjective.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

*For full credit, please write out fully what you mean instead of using shorthand phrases.

(a) If A is the standard matrix of a linear transformation $T : \mathbb{R}^{71} \to \mathbb{R}^{71}$, then

$$\ker(T) \subset \ker(T \circ T).$$

> **Solution:** TRUE. If $x \in \ker(T)$, then $T(x) = 0$. Hence $(T \circ T)(x) = T(T(x)) = T(0) = 0$, so $x \in \ker(T \circ T)$.

(b) For all matrices A and B for which the products AB and BA are both defined, if $AB = 0$ then also $BA = 0$.

> **Solution:** FALSE. Counterexample: let
>
> $$A = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (1 \times 2), \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (2 \times 1).$$
>
> Then $AB = [1\ 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$ (the 1×1 zero matrix), but
>
> $$BA = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq 0.$$
>
> Thus $AB = 0$ does not imply $BA = 0$.